

Homework 3

Math 121A, Fall 2025.

Due: Wednesday, Sep 17, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. For each of the following subsets in $\mathbb{R}^3$, decide whether it is a subspace of $\mathbb{R}^3$:

- (a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
- (b) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
- (c) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1x_2x_3 = 0\}$;
- (d) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 5x_3\}$.

Problem 2. (MAT 54 review 1)

- (a) Consider the system of linear equations

$$\begin{cases} z - y + w = 0 \\ x + y = 2 \\ x - y = z + w. \end{cases}$$

Rewrite this as a linear equation of the form $M\vec{x} = \vec{b}$, where $\vec{x} = (x \ y \ z \ w)^T$.

- (b) We may think of M as defining a linear map $M : \mathbb{R}^n \rightarrow \mathbb{R}^m$, after choosing the standard basis for $\mathbb{R}^n$ and for $\mathbb{R}^m$. What are n and m here? (*note that it helps to think of input and output dimensions*).

Before proceeding, we recall two key definitions from MAT 54. Given a linear map $M : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the *range* $\text{ran}(M)$ and *null space* $\text{null}(M)$ are the subspaces defined by

$$\begin{aligned} \text{ran}(M) &:= \{\vec{b} \in \mathbb{R}^m : \text{there exists a solution } \vec{x} \text{ to } M\vec{x} = \vec{b}\} \\ \text{null}(M) &:= \{\vec{x} \in \mathbb{R}^n : M\vec{x} = 0\}. \end{aligned}$$

Note that $\text{ran}(M)$ is a subspace of $\mathbb{R}^m$, while $\text{null}(M)$ is a subspace of $\mathbb{R}^n$. The *rank* of M is then $\text{rank}(M) = \dim \text{ran}(M)$.

- (c) (*Existence of a solution*) Find a solution $\vec{x}_p$ to the equation $M\vec{x} = \vec{b}$ using Gaussian elimination.¹ Is $\vec{b} \in \text{ran}(M)$?

¹This is called the *particular* solution to the *inhomogeneous* equation.

(d) (*Uniqueness of a solution*)

Recall a useful fact: a solution to $M\vec{x} = \vec{b}$ is unique if and only if $\dim \text{null}(M) = 0$.

By computing the null space of M , decide whether the solution from part (c) is unique.

(*Hint: the null space can be read off from the RREF of M you computed earlier.*)

(e) (*Dimension of solution space*) The solution space $S := \{\vec{x} \in \mathbb{R}^n : M\vec{x} = \vec{b}\}$ may be thought of as the set of “particular + homogeneous solutions”, i.e. $\vec{x}_p$ solves $M\vec{x} = \vec{b}$ from part (c) and

$$S = \{\vec{x}_p + \vec{x}_h : \vec{x}_h \in \text{null}(M)\}.$$

What is the dimension of S ? (*Note: S is not a vector space, but it is a vector space shifted by $\vec{x}_p$ and so we may talk about its dimension. Don't overthink this one.*)

(f) Compute $\text{rank}(M)$ and deduce the following:

i. The equation $M\vec{x} = \vec{c}$ has a solution for any vector $\vec{c} \in \mathbb{R}^m$.

ii. Let $S_c := \{\vec{x} \in \mathbb{R}^n : M\vec{x} = \vec{c}\}$ denote the solution space for this equation. Then $\dim(S_c) = \dim(S)$.

This does not require you to re-do any work!

Problem 3. (*MAT 54 review 2*)

(a) Consider the system of linear equations

$$\begin{cases} x + y = b_1 \\ x - y = b_2 \\ 2x - 3y = b_3. \end{cases}$$

Rewrite this as a linear equation of the form $M\vec{x} = \vec{b}$, where $\vec{x} = \begin{pmatrix} x & y \end{pmatrix}^T$ and $\vec{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$.

(b) We may think of M as defining a linear map $M : \mathbb{R}^n \rightarrow \mathbb{R}^m$, after choosing the standard basis for $\mathbb{R}^n$ and for $\mathbb{R}^m$. What are n and m here?

(c) Without computing anything, answer the following question: is it possible to have a solution for every $\vec{b} \in \mathbb{R}^3$?

(d) Find the rref of M and compute $\text{rank}(M)$ and $\dim \text{null}(M)$.

Problem 4. (*Change of coordinates*) In physics we often need to describe change of coordinates which make problems more tractable. Linear ones are among the most important.

Take the standard basis of $\mathbb{R}^3$, i.e. $\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Write the 3×3 matrix $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which...

(a) ...stretches by 2 in the z direction and does nothing in the other directions, i.e. $T\hat{i} = \hat{i}$, $T\hat{j} = \hat{j}$, and $T\hat{k} = 2\hat{k}$.

(b) ...shrinks the x direction by $1/2$ and does nothing in the other directions.

(c) ...swaps the x coordinate and the y coordinate.

(d) ...rotates the xy plane counterclockwise around the z axis by an angle θ .

(e) ...reflects the space by mapping $\hat{k} \mapsto -\hat{k}$.

Problem 5. (*Rotations*) Let's study the rotation matrix a little more carefully. Let $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear map which rotates the xy plane counterclockwise by an angle $\theta \in [0, 2\pi)$.

- (a) Take the standard basis of $\mathbb{R}^2$ and write the matrix $[R_\theta]$.
- (b) Fix an arbitrary angle θ . Where does R_θ send the standard basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?
- (c) Compute the inverse of R_θ and interpret its action geometrically.

Problem 6. (*Our first quantum state*) The following matrix describes a famous single-qubit quantum state in superposition:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

We may think of this as the matrix of linear map $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ in the standard basis.

- (a) Compute bases for $\text{ran}(\rho)$ and $\text{null}(\rho)$ and normalize them (i.e. make each vector have $\|v\| = 1$). What are the dimensions of these spaces?
- (b) Let $\mathcal{B} = \{v_1, v_2\}$ be a basis of $\mathbb{R}^2$ consisting of vectors from part (a). Write the matrix $[\rho]_{\mathcal{B}}$ in this basis.
- (c) Expand a vector $\psi \in \mathbb{R}^2$ in the basis $\mathcal{B}$, i.e. $\psi = \psi_1 v_1 + \psi_2 v_2$. What is $\rho(\psi)$? Interpret this geometrically.

Problem 7. (*Function spaces*) Many physical systems can be modeled using vector spaces of functions. These often encode physical observables, like the position or momentum of a particle on a wire. Consider the vector space of polynomials $p : [0, 1] \rightarrow \mathbb{R}$ given by

$$V = \text{span}\{1, x, x^2\}.$$

- (a) Show that $\{1, x, x^2\}$ is a basis for V (this requires showing that these vectors are linearly independent, which may be done by judiciously picking 3 x values and verifying independence, or by checking the Wronskian).
- (b) Express the vector $p(x) = 1 - 2x + 3x^2$ in this basis.
- (c) We know from calculus that the derivative operator is linear. Show that the derivative operator $\frac{\partial}{\partial x}$ is a linear map from $V \rightarrow V$, and calculate the matrix $[\frac{\partial}{\partial x}]$ in the listed basis.
- (d) Is the derivative operator $\frac{\partial}{\partial x} : V \rightarrow V$ invertible?
- (e) We also know from calculus that integration yields a linear map. Consider $I : V \rightarrow \mathbb{R}$ given by²

$$I(p) = \int_0^1 p(x) dx.$$

Compute the matrix corresponding to I (using the standard basis 1 of $\mathbb{R}$).

²These types of maps are called *linear functionals*, and they represent physical states.